Ming Hsieh Department of Electrical Engineering

Identifiability Results for Ill-posed Bilinear Inverse Problems

USC Viterbi School of Engineering

Sunav Choudhary and Urbashi Mitra

Communication Sciences Institute

Main Message

- Identifiability crucial in inverse problems
  - Not well understood for non-linear systems/constraints
- We develop theory for Bilinear Inverse Problems
  - subsumes blind estimation
  - deterministic characterization of identifiability
  - probabilistic scaling law
  - general conic constraints included, e.g. sparsity and low rank constraints
- Connect blind estimation to low-rank matrix recovery
  - readily available convex relaxations

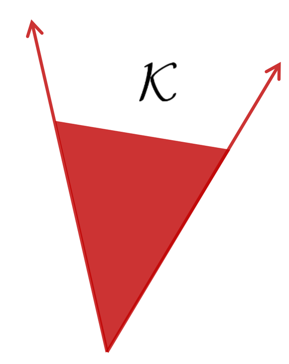
Introduction

Given $\boldsymbol{z} = \boldsymbol{S}(\boldsymbol{x},\boldsymbol{y})$ (Bilinear Map)

$$\begin{aligned} &\text{Find } (\boldsymbol{x},\boldsymbol{y}) \\ &\text{Subject to } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{K} \end{aligned}$$

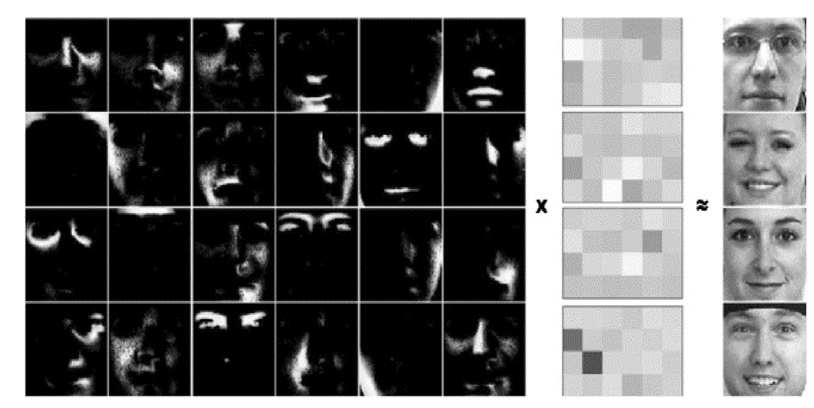
$\boldsymbol{S}(\cdot,\boldsymbol{y})$ linear $\forall \boldsymbol{y} \in \mathbb{R}^n$

$\boldsymbol{S}(\boldsymbol{x},\cdot)$ linear $\forall \boldsymbol{x} \in \mathbb{R}^m$



$\mathcal{K}$ is a cone

Matrix Factorization



Given $\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{Y}$ (Bilinear Map)

$$\begin{aligned} &\text{Find } (\boldsymbol{X},\boldsymbol{Y}) \\ &\text{Subject to } X_{ij}, Y_{jk} \geq 0 \end{aligned}$$

(Conic Constraint)

Lifting

$$\begin{aligned} &\text{Find} && (\boldsymbol{x},\boldsymbol{y}) \\ &\text{Subject to} && \boldsymbol{S}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{z} \\ & && (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{K} \end{aligned}$$

$$\begin{aligned} &\underset{\boldsymbol{W}}{\text{minimize}} && \operatorname{rank}(\boldsymbol{W}) \\ &\text{subject to} && \mathscr{S}(\boldsymbol{W}) = \boldsymbol{z} \\ & && \boldsymbol{W} \in \mathcal{K}' \end{aligned}$$

$\phi_k \circ \boldsymbol{S}(\boldsymbol{x},\boldsymbol{y}) = z_k$ (ϕ_k: k^{th} coordinate projector)

$$\phi_k \circ \boldsymbol{S}(\boldsymbol{x},\boldsymbol{y}) = \langle \boldsymbol{x}, \boldsymbol{S}_k \boldsymbol{y} \rangle = \langle \boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}, \boldsymbol{S}_k \rangle$$

Inner Product in $\mathbb{R}^m$

Trace Inner Product in $\mathbb{R}^{m \times n}$

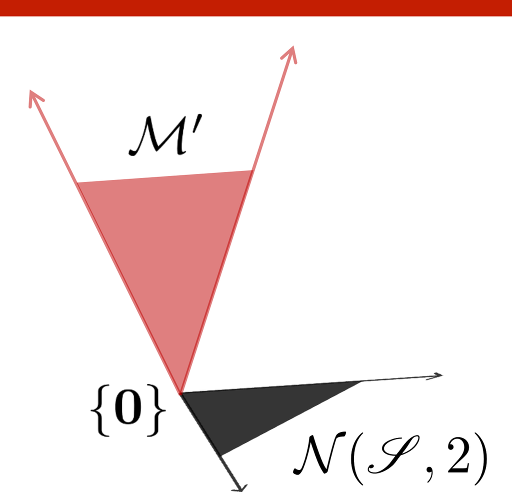
Linear Convolution: $\boldsymbol{S}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x} \star \boldsymbol{y}$

$(m = 3, n = 4, p = m + n - 1 = 6)$

$$S_1 = \begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix},\quad S_2 = \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix},\quad S_3 = \begin{bmatrix} 0&0&1&0\\0&1&0&0\\1&0&0&0 \end{bmatrix}$$

$$S_4 = \begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0 \end{bmatrix},\quad S_5 = \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix},\quad S_6 = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}$$

Universal Identifiability

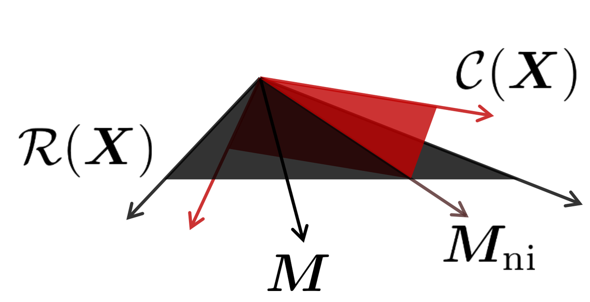


$\mathcal{M}'$ is domain of ambiguity

$\mathcal{M}' = \{\boldsymbol{Y} - \boldsymbol{Z} \mid \boldsymbol{Y},\boldsymbol{Z} \in \mathcal{K}'\}$

$\mathcal{N}(\mathscr{S},2)$ is rank-2 null space

Instance Identifiability



$\boldsymbol{M}$ is identifiable

$\boldsymbol{M}_{\text{ni}}$ is *not* identifiable

$\boldsymbol{X}$ in rank-2 null space

$\mathcal{R}(\cdot)$ is row space

$\mathcal{C}(\cdot)$ is column space

Exponential Scaling Law

- i.i.d. Gaussian/Bernoulli Inputs
- Probability of Identifiability =

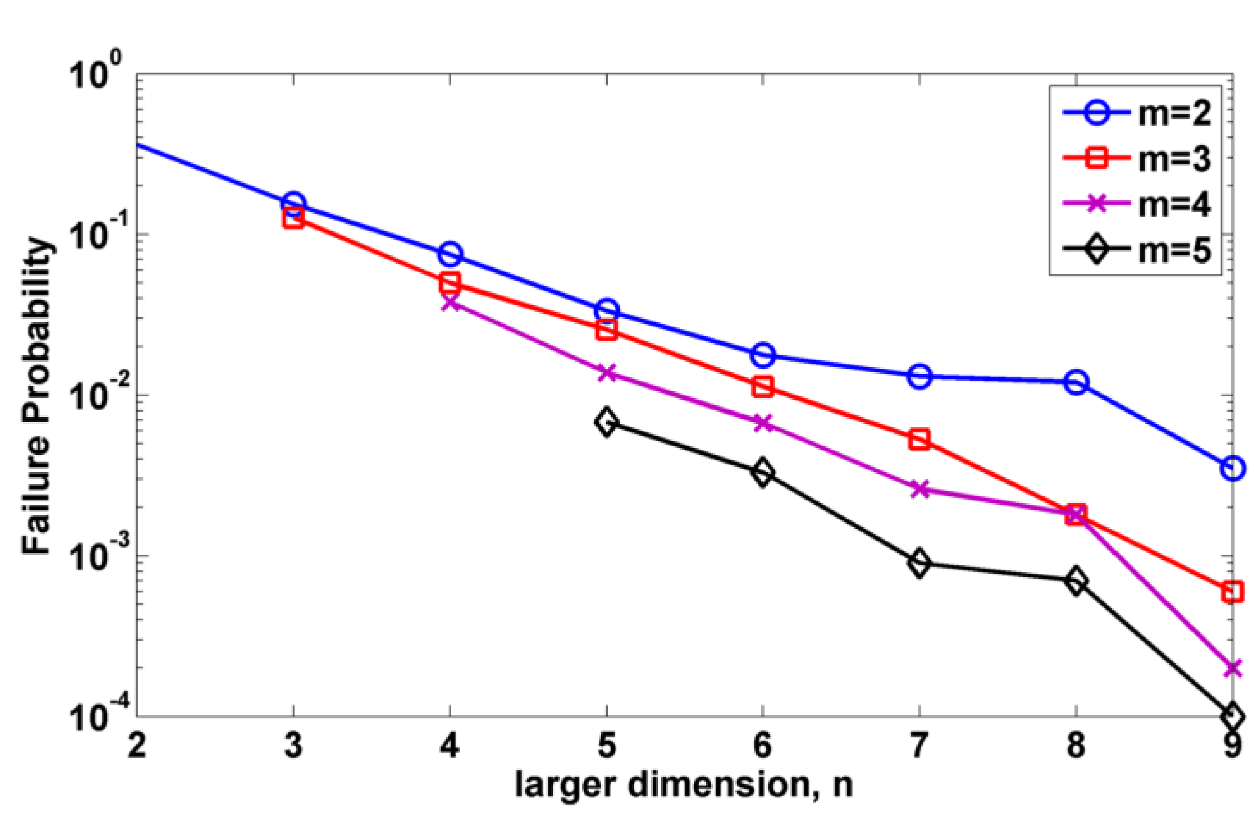
$$1 - \exp\left[C_1 \cdot p - C_2 \cdot (m+n)\right]$$

- p is DoF in rank-2 null space
- m, n are problem dimensions
- $p = o(m+n)$ implies identifiability w.h.p.

Simulation Results

$$\begin{aligned} &\underset{\boldsymbol{X}}{\text{minimize}} && \operatorname{rank}(\boldsymbol{X}) \\ &\text{subject to} && \|\boldsymbol{X} - \boldsymbol{M}\|_{\mathrm{F}} \leq \epsilon \\ & && \mathscr{S}(\boldsymbol{X}) = \boldsymbol{0} \end{aligned}$$

- Used Reweighted Nuclear Norm Heuristic
- Used Convolution Operator



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email: s.choudhary@usc.edu

Web: http://www-scf.usc.edu/~sunavcho/

Ming Hsieh Institute

Ming Hsieh Department of Electrical Engineering