# Target Positioning Using TOA Measurements

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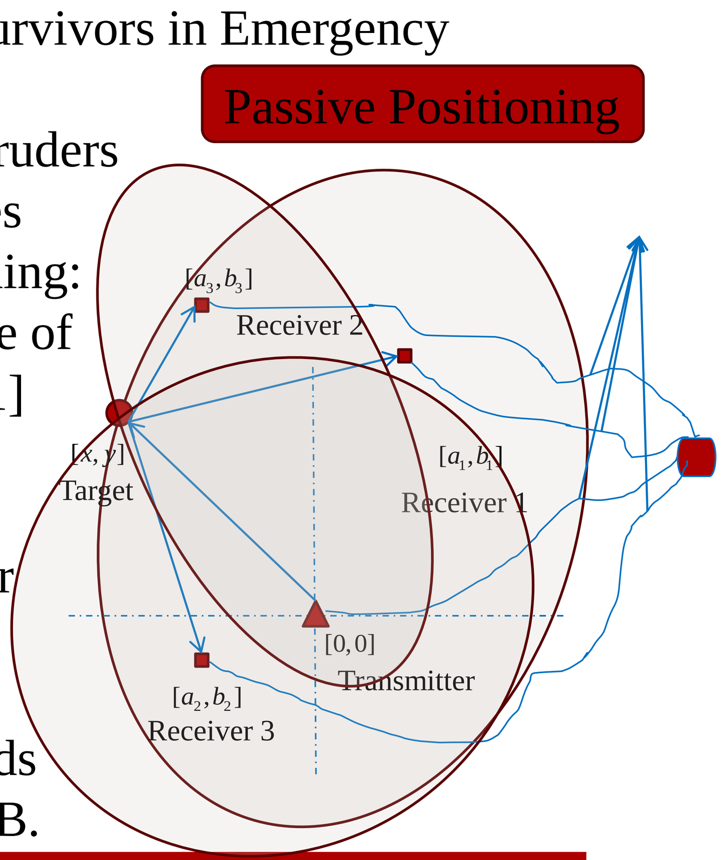
## Introduction

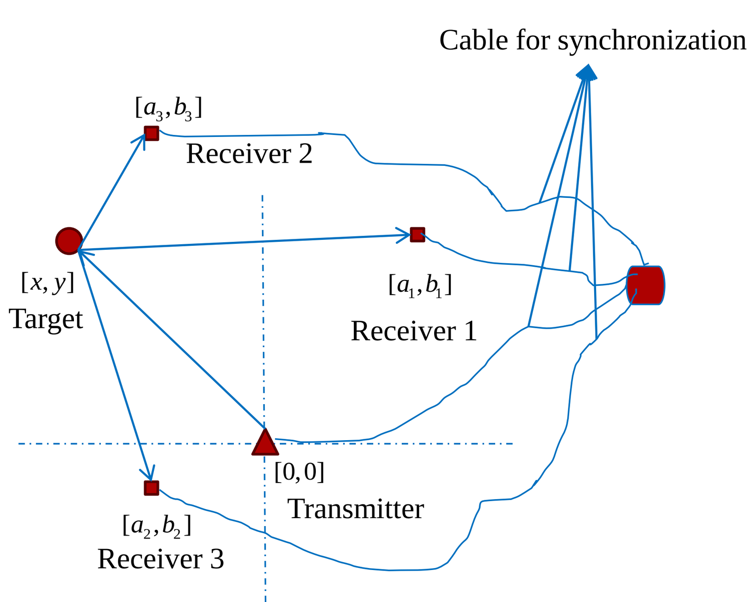
❖Radio Positioning can be divided into
1. Active Target: target transmits signals
2. Passive Target: target reflects signals

❖Applications of Passive Radio Positioning
•Localization of Survivors in Emergency Rescue
•Positioning of Intruders

❖ Previous researches on passive positioning:
1. Time difference of arrival (TDOA): [1] proposes a method which approaches Cramer-Rao Lower Bound (CRLB)
2. Time of arrival (TOA): No methods can approach CRLB.




## Problem & Hypothesis




❖Known Information
1. Locations of transmitter and receivers,
2. Signal travel ranges from the transmitter to receivers.

❖Objective
❑Estimate the target location denoted by $[\hat{x},\hat{y}]$.

❖Estimation Criteria
❑Minimum Square Error
$$\Delta = E[(\hat{x}-x)^2+(\hat{y}-y)^2]$$

## Two Step Expectation Maximization (TSEM) Algorithm --- Step 1

➢The measured signal travel ranges at the receivers are

$$r_i=\sqrt{x^2+y^2}+\sqrt{(x-a_i)^2+(y-b_i)^2}\quad i=1,\ldots,M. \qquad (1)$$

➢ With some manipulations, (1) can be rewritten as

$$2a_ix+2b_iy-2r_i\sqrt{x^2+y^2}=a_i^2+b_i^2-r_i^2 \qquad (2)$$

➢Assume the range measurements are Gaussian distributed

$$r_i=\hat{r}_i+e_i,\qquad 1\le i\le M, \qquad (3)$$

➢ Substitute (3) into (2), we obtain

$$-\frac{a_i^2+b_i^2-\hat{r}_i^2}{2}+a_ix+b_iy-\hat{r}_i\sqrt{x^2+y^2}$$
$$=e_i(\sqrt{x^2+y^2}-\hat{r}_i)-\frac{e_i^2}{2}. \qquad (4)$$

For small error $e_i$, we can omit the second order items.

➢Since there are $M$ equations like (4), they can be formulated in a matrix form as follows

$$\mathbf{h}-\mathbf{S}\boldsymbol{\theta}=\mathbf{B}\mathbf{e}, \qquad (5)$$

where

$$\mathbf{h}=\begin{bmatrix}-\frac{a_1^2+b_1^2-\hat{r}_1^2}{2}\\ -\frac{a_2^2+b_2^2-\hat{r}_2^2}{2}\\ \vdots\\ -\frac{a_M^2+b_M^2-\hat{r}_M^2}{2}\end{bmatrix},\quad \mathbf{S}=-\begin{bmatrix}a_1 & b_1 & -\hat{r}_1\\ a_2 & b_2 & -\hat{r}_2\\ & \vdots & \\ a_M & b_M & -\hat{r}_M\%\end{bmatrix},$$

$$\boldsymbol{\theta}=[x,y,\sqrt{x^2+y^2}]^T,\qquad \mathbf{e}=[e_1,e_2,\ldots,e_M]^T.$$

➢ For notational convenience, we define an error vector

$$\boldsymbol{\varphi}=\mathbf{h}-\mathbf{S}\boldsymbol{\theta}.$$

➢The covariance matrix of $\boldsymbol{\varphi}$ is given as

$$\boldsymbol{\Psi}=E(\boldsymbol{\varphi}\boldsymbol{\varphi}^T)=\mathbf{B}\mathbf{Q}\mathbf{B}^T. \qquad (6)$$

where $\mathbf{Q}=\text{diag}[\sigma_1^2,\sigma_2^2,\ldots,\sigma_M^2],\sigma_i^2$ is the range measurement error at the $i$th receiver.

➢ The probability density function of $\boldsymbol{\varphi}$ is

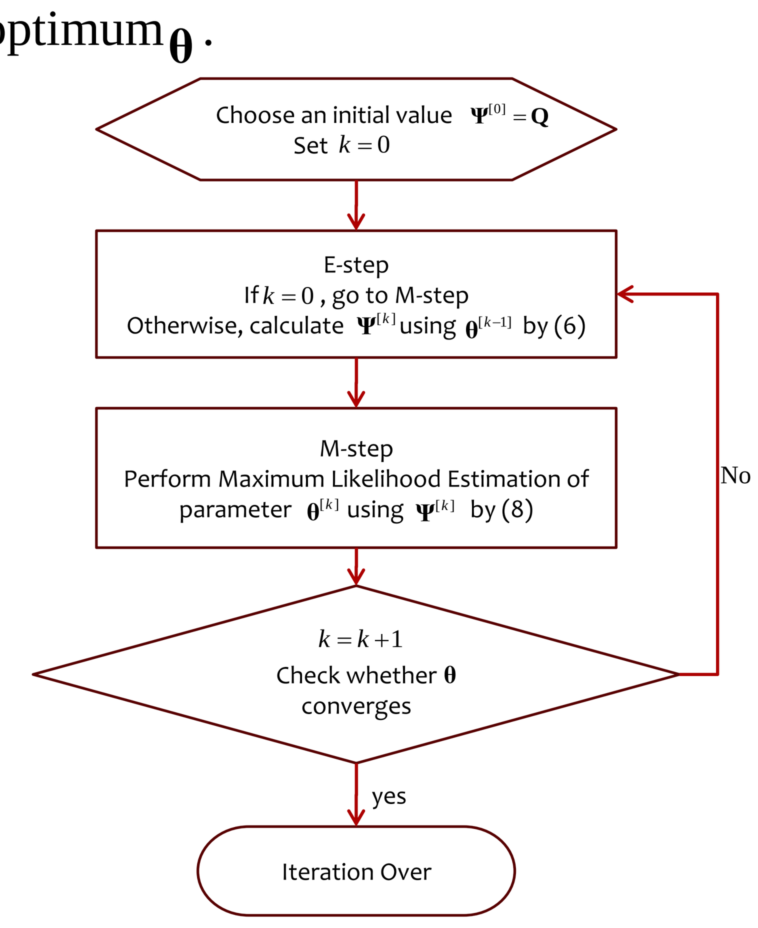
$$p(\boldsymbol{\varphi}\mid\boldsymbol{\theta})=\frac{1}{(2\pi)^{\frac{M}{2}}\,|\boldsymbol{\Psi}|^{\frac{1}{2}}}\exp(-\frac{1}{2}\boldsymbol{\varphi}^T\boldsymbol{\Psi}^{-1}\boldsymbol{\varphi}) \qquad (7)$$
$$=\frac{1}{(2\pi)^{\frac{M}{2}}\,|\boldsymbol{\Psi}|^{\frac{1}{2}}}\exp(-\frac{1}{2}(\mathbf{h}-\mathbf{S}\boldsymbol{\theta})^T\boldsymbol{\Psi}^{-1}(\mathbf{h}-\mathbf{S}\boldsymbol{\theta})).$$

➢ If we assume that $x$, $y$ and $\sqrt{x^2+y^2}$ are independent with each other, and $\boldsymbol{\Psi}$ is a constant, the optimum $\boldsymbol{\theta}$ satisfies

$$\boldsymbol{\theta}=(\mathbf{h}-\mathbf{S}\boldsymbol{\theta})^T\boldsymbol{\Psi}^{-1}(\mathbf{h}-\mathbf{S}\boldsymbol{\theta})+\ln|\boldsymbol{\Psi}|)$$
$$=(\mathbf{S}^T\boldsymbol{\Psi}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\boldsymbol{\Psi}^{-1}\mathbf{h} \qquad (8)$$

However, for our problem, $\boldsymbol{\Psi}$ is a function of $\boldsymbol{\theta}$.

➢ Thus, we employ the Expectation Maximization (EM) algorithm to find the optimum $\boldsymbol{\theta}$.




## Two Step Expectation Maximization (TSEM) Algorithm --- Step 2

❖Construct a vector $\mathbf{g}$ as follows

$$\mathbf{g}=\boldsymbol{\Theta}-\mathbf{G}\Upsilon \qquad (9)$$

where $\boldsymbol{\Theta}=[\hat{x}^2,\hat{y}^2,\hat{x}^2+\hat{y}^2]^T, \Upsilon=[x^2,y^2]^T$

and $\mathbf{G}=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}$.

➢ The optimum $\Upsilon$ is the one minimizing $(\boldsymbol{\Theta}-\mathbf{G}\Upsilon)^T\boldsymbol{\Omega}^{-1}(\boldsymbol{\Theta}-\mathbf{G}\Upsilon)$, denoted by

$$\Upsilon=(\mathbf{G}^T\boldsymbol{\Omega}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\boldsymbol{\Omega}^{-1}\Theta. \qquad (10)$$

❖Finally, the estimate of target location $[x,y]$ is obtained by

$$\hat{\mathbf{z}}=[\hat{x},\hat{y}]=[\pm\sqrt{\Upsilon_1},\pm\sqrt{\Upsilon_2}]. \qquad (11)$$

where $\Upsilon_i$ is the $i$th item of $\Upsilon$.

❖The final result is the one of the four options in (13) minimizing the square error as follows

$$\chi=\sum_{i=1}^{M}(\sqrt{\hat{x}^2+\hat{y}^2}+\sqrt{(\hat{x}-a_i)^2+(\hat{y}-b_i)}-\hat{r}_i)^2. \qquad (12)$$

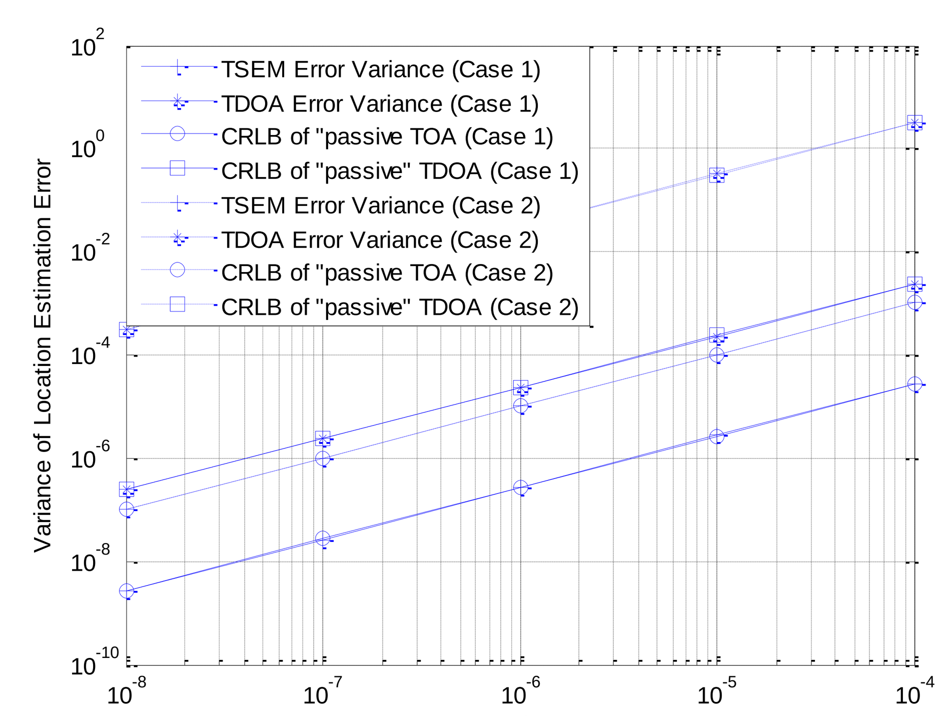
## TSEM Algorithm

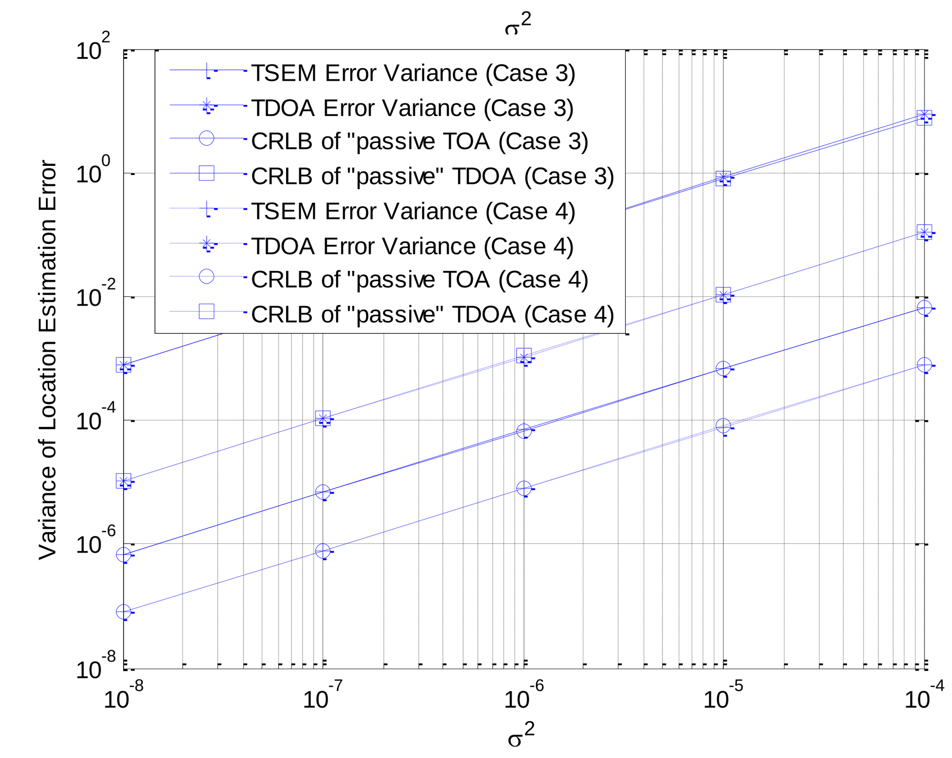
1. Use EM algorithm to obtain $\theta$ ,
2. Use the values of $x$ and $y$ , generate $\boldsymbol{\Theta}$ and $\mathbf{D}$ , and calculate $\boldsymbol{\Omega}$ . Then, calculate the value of $\Upsilon$ according to (10),
3. Among the four potential values of obtained by (11), choose the one minimizing (12) as the final estimate for target location .

## Simulation Results

Table I
SIMULATION PARAMETERS

| | Target Location | Receivers Locations (Error Variances) |
|---|---|---|
| Configuration 1 | [3, 8] | [-1 1] ($0.1\sigma^2$), [2 1] ($0.13\sigma^2$), [-3, 1.1] ($0.12\sigma^2$), [4 0] ($0.095\sigma^2$) |
| Configuration 2 | [31, 28] | [-1 1] ($0.1\sigma^2$), [2 1] ($0.13\sigma^2$), [-3, 1.1] ($0.12\sigma^2$), [4 0] ($0.095\sigma^2$) |
| Configuration 3 | [10, 13] | [-1 1] ($\sigma^2$), [1 1] ($\sigma^2$), [1, -1] ($\sigma^2$), [-1, -1] ($\sigma^2$) |
| Configuration 4 | [10, 13] | [-3 3] ($\sigma^2$), [3 3] ($\sigma^2$), [3, -3] ($\sigma^2$), [-3, -3] ($\sigma^2$) |




It can be observed that
1) The localization error of TSEM can closely approach the CRLB of TOA based positioning algorithms.
2) The CRLB (minimum error variance) of TDOA based positioning algorithm in [1] is much higher (about 20dB) than that of TOA based.

1. Y.~Chan and K.~Ho, ``A simple and efficient estimator for hyperbolic location,'' *IEEE Transactions on Signal Processing*, vol.~42, no.~8, pp. 1905--1915, 1994.