Quantum-Jump Continuous-Time Quantum Error-Correction

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Motivation

- Continuous-time quantum error-correction (CTQEC) is a study of protecting quantum information against decoherence, where both the decoherence and error-correction processes are considered continuous in time.
- Quantum-jump CTQEC is a model for CTQEC that describes a quantum system with a continuous master equation. One important task in concerning this model is to find a protocol that realizes the quantum-jump CTQEC mapping

$$\rho \xrightarrow{\delta t \ll 1} \Big(1 - \underbrace{\boxed{\kappa\delta t}}_{\varepsilon}\Big)\,\rho + \underbrace{\boxed{\kappa\delta t}}_{\varepsilon}\,\mathcal{R}(\rho), \quad (1)$$

where $\mathcal{R}(\cdot)$ is a recovery mapping.

Methods and Approach

- Consider an $[[n,k,d]]$ stabilizer code.
- *Lemma 1* $\exists$ *a corrected basis in the full Hilbert space $\mathcal{H}^{\otimes n}$ in which correctable errors of the code only flip the qubits in $\mathcal{H}_S$, the "syndrome" system of $n-k$ qubits, and leave the qubits in $\mathcal{H}_I$, the "information" system of k qubits, invariant.*
- In the corrected basis,

$$\mathcal{R}(\rho) = \sum_{j=0}^{2^{n-k}-1} R_j \rho R_j^\dagger \quad (2)$$

where $R_j = I^{\otimes k} \otimes |\underline{0}\rangle\langle \underline{j}|, \forall j$.

- Finding set of Kraus operators, $\{K_l\}_{l\in\mathbb{Z}_S}$, for mapping (1) is important in our derivation of CTQEC protocols. Specifically, we need $K_l \propto$ a weak operator.

$$\sum_{l\in\mathbb{Z}_S} K_l \rho K_l^\dagger = (1-\varepsilon)\,\rho + \varepsilon \sum_{j=0}^{2^{n-k}-1} R_j \rho R_j^\dagger \quad (3)$$

- "Polar decomposition" decomposes each K_l into

$$K_l = U_l M_l, \quad (4)$$

  - U_l is unitary.
  - M_l is positive-semidefinite.
  - $\{M_l\}_{l\in\mathbb{Z}_S}$ a POVM measurement.
- Evidently,
  - In *Step IV*, $U_{c,l} = U_l, \forall l \in \mathbb{Z}_S$.
  - *Steps I-III* realizes POVM $\{M_l\}_{l\in\mathbb{Z}_S}$.
  - $s_a \geq \lceil log_2 S\rceil$. ($= \lceil log_2 S\rceil$ sufficient)
  - If $S = 2^c$ for some $c \in \mathbb{N}$, $s_a = c$,

$$U_M\left(|\psi\rangle \otimes |A_0\rangle\right) = \sum_{l\in\mathbb{Z}_S} M_l|\psi\rangle \otimes |\underline{l}_a\rangle. \quad (5)$$

- Trick to find U_M:
  - Let $U_M = e^{i\varepsilon H}$ and Hermitian $H = \sum_{l,m\in\mathbb{Z}_S} H_{l,m} \otimes |\underline{l}_a\rangle\langle \underline{m}_a|$.
  - Expand U_M to order ε^2, apply (5).

Contribution

- For any given quantum stabilizer code, we found a set of criterion that leads to a large family of protocols that realize the corresponding quantum-jump CTQEC mapping.
- We propose a CTQEC protocol, based on any given $[[n,k,d]]$ quantum stabilizer code, that requires only $n-k+1$ ancillary qubits in the quantum measurement phase. This greatly reduces the number of ancillary qubits required from 2^{n-k} as in [1].
- The comparison of existing CTQEC protocols.

Typical Procedures in a CTQEC protocol

- *Step I*: Couple an ancillary system of size s_a, prepared in the state $|A_0\rangle = |+\rangle^{\otimes s_a} = \sum_{l=0}^{2^{s_a}-1} |\underline{l}_a\rangle$, to the code system.
- *Step II*: Apply a weak[1] unitary U_M to the combined system.
- *Step III*: Measure each ancillary qubit in the standard $\{|0\rangle, |1\rangle\}$ basis.
- *Step IV*: Apply a weak[1] unitary correction, $U_{C,m}$, conditioned on measurement outcome m.

[1] Weak operator has the form $\mathbb{I} + \underline{\mathcal{O}}(\varepsilon)$, where $\mathbb{I}$ is the identity operator and $\underline{\mathcal{O}}(\varepsilon)$ has matrix elements in the order $\mathcal{O}(\varepsilon)$

Theorem: Quantum-Jump CTQEC with Minimal, $n-k+1$, Ancillas

- $s_a = n-k+1$. $S = 2^{n-k+1}$, and $\forall l \in \{0,1,\ldots,2^{n-k}-1\}$,
  - $K_l = \frac{1}{\sqrt{2^{n-k+1}}} I^{\otimes k} \otimes \left(\sqrt{1-\varepsilon^2}\, I^{\otimes n-k} + i\varepsilon\sqrt{2^{n-k}}|\underline{0}\rangle\langle\underline{l}|\right)$,
  - $K_{2^{n-k}+l} = \frac{1}{\sqrt{2^{n-k+1}}} I^{\otimes k} \otimes \left(\sqrt{1-\varepsilon^2}\, I^{\otimes n-k} - i\varepsilon\sqrt{2^{n-k}}|\underline{0}\rangle\langle\underline{l}|\right)$.
- $U_M = e^{i\varepsilon H}$, and $H = \sum_{l,m=0}^{2^{n-k+1}} H_{l,m} \otimes |\underline{l}_a\rangle\langle\underline{m}_a|$.
  - $H_{0,0} = -H_{2^{n-k},2^{n-k}} = \frac{-1}{\sqrt{2^{n-k-2}}} I^{\otimes k} \otimes \sum_{l=1}^{2^{n-k}-1} (|\underline{0}\rangle\langle\underline{l}| + |\underline{l}\rangle\langle\underline{0}|)$,
  - $H_{0,2^{n-k}} = H_{2^{n-k},0} = 0$,

  and $\forall\, l \in \{1,2,\ldots,2^{n-k}-1\}$,
  - $H_{0,l} = H_{l,0} = \frac{1}{\sqrt{2^{n-k-2}}} I^{\otimes k} \otimes (|\underline{0}\rangle\langle\underline{l}| + |\underline{l}\rangle\langle\underline{0}|)$,
  - $H_{0,2^{n-k}+l} = H_{2^{n-k}+l,0} = 0$,
  - $H_{l,l} = -H_{2^{n-k}+l,2^{n-k}+l} = \frac{1-2^{n-k}}{\sqrt{2^{n-k-2}}} I^{\otimes k} \otimes (|\underline{0}\rangle\langle\underline{l}| + |\underline{l}\rangle\langle\underline{0}|)$,
  - $H_{2^{n-k},l} = H_{l,2^{n-k}} = 0$,
  - $H_{2^{n-k},2^{n-k}+l} = H_{2^{n-k}+l,2^{n-k}} = -\frac{1}{\sqrt{2^{n-k-2}}} I^{\otimes k} \otimes (|\underline{0}\rangle\langle\underline{l}| + |\underline{l}\rangle\langle\underline{0}|)$,
  - $H_{l,2^{n-k}+l} = -H_{2^{n-k}+l,l} = \sqrt{2^{n-k-2}} I^{\otimes k} \otimes (|\underline{0}\rangle\langle\underline{l}| - |\underline{l}\rangle\langle\underline{0}|)$,

  and $\forall\, l,m \in \{1,2,\ldots,2^{n-k}-1\}, l \neq m$,
  - $H_{l,m} = -H_{2^{n-k}+l,2^{n-k}+m} = \frac{1}{\sqrt{2^{n-k}}} I^{\otimes k} \otimes (|\underline{0}\rangle\langle\underline{l}| + |\underline{l}\rangle\langle\underline{0}| + |\underline{0}\rangle\langle\underline{m}| + |\underline{m}\rangle\langle\underline{0}|)$,
  - $H_{l,2^{n-k}+m} = -H_{2^{n-k}+l,m} = \frac{1}{\sqrt{2^{n-k}}} I^{\otimes k} \otimes (|\underline{0}\rangle\langle\underline{l}| + |\underline{l}\rangle\langle\underline{0}| - |\underline{0}\rangle\langle\underline{m}| - |\underline{m}\rangle\langle\underline{0}|)$.
- For all $l \in \{0,1,2,\ldots,2^{n-k}-1\}$, $U_{C,l} = e^{iH_{C,l}}$, where
  - $H_{C,l} = -H_{C,2^{n-k}+l} = \varepsilon\frac{\sqrt{2^{n-k}}}{2}(|\underline{0}\rangle\langle\underline{l}| + |\underline{l}\rangle\langle\underline{0}|)$.

Near-Future Work

- The comparison between CTQEC protocols involve choosing "fair" measurement and correction strengths. Our simulation using the three-qubit code shows a slight worse in performance comparing to the protocol in [1], maybe due to the trade-off from the size of ancillas. However, simulation shows great performance gain conparing with the protocols in [2].

References

[1] O. Oreshkov, "Continuous-time quantum error correction," (2013), quant-ph/1311.2485.
[2] C. Ahn, A. Doherty, and A. Landahl, "Continuous quantum error correction via quantum feedback control," *Phys. Rev. A* 65 (2002).